

Groups from Class 2 Algebras and the Weil Character

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Abstract

We investigate the behaviour of the Weil character of the symplectic group on restriction to subgroups arising from commutative nilpotent algebras of class 2. We give explicit descriptions of the decomposition of the Weil character when restricted to the unipotent radical of the stabilizer of a maximal totally isotropic subspace and to its centralizer.

1 Introduction

Because of their significant properties, Weil representations play an important role in the study of the representation theory of classical groups. The characters of Weil representations have been computed by various authors; see for example [1], [5], [6], [11]. We will be making use of some explicit results concerning the Weil characters of the symplectic group as these are obtained in [14] via the ‘theta form’ (see also [11]). The approach in [14] is to follow the treatment for the Weil representation given in [13].

One of the aims of the present work is to follow up in the direction of some of the investigations in [9], [10] concerning the restriction of the Weil characters of symplectic and unitary groups to certain subgroups, in particular to certain self-centralizing subgroups. In [10] an explicit description of the restriction of the Weil character (respectively, the sum of the two Weil characters) of the unitary group (respectively, symplectic group) to centralizers of regular unipotent elements was also obtained. We remark here that the study of the behaviour of element centralizers in Weil representations already began in [4].

At this point we introduce some notation. Fix q a power of an odd prime and consider the Weil representation and its character ω for $\mathrm{Sp}(2n, q)$, regarded as a matrix group on $V = GF(q)^{2n}$ defined via the symplectic form φ having matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. The subgroup G

of $\mathrm{Sp}(2n, q)$ to be considered is $G = \langle -I \rangle \times B$ where B is the set of matrices $g_Q = \begin{bmatrix} I & 0 \\ Q & I \end{bmatrix}$, with Q a symmetric matrix. So B is the unipotent radical of the stabilizer in $\mathrm{Sp}(2n, q)$ of a maximal totally isotropic subspace of V , and G , which is the centralizer of B in $\mathrm{Sp}(2n, q)$, is a maximal Abelian subgroup of $\mathrm{Sp}(2n, q)$. Also note that the matrices $g_Q - I$ with $g_Q \in B$ belong to a nilpotent subalgebra of the full matrix algebra of $2n \times 2n$ matrices over $GF(q)$ of class 2 (compare with the discussion in [9]).

The restriction $\omega|_G$ is known to be multiplicity free. The main result of the paper is Theorem 5, where we describe explicitly the irreducible characters of G (respectively, B) appearing in the decomposition of $\omega|_G$ (respectively, $\omega|_B$). As a by-product, in Section 7 we use the expression for the decomposition $\omega|_B$ in order to obtain alternative derivations for the number of solutions of the equation $Q(x) = \alpha$ with $\alpha \in GF(q)$, $x \in GF(q)^n$ and Q a quadratic form of $GF(q)^n$.

One can easily observe that a subgroup H of G with a multiplicity free restriction $\omega|_H$ satisfies $|H| \geq 2q^n$. In Section 9 we show that this bound is actually attained and we give a construction of such a subgroup H with $|H| = 2q^n$ and $\omega|_H$ being multiplicity free. Finally, we show how an evaluation concerning the values of the Weil character on the elements of this subgroup H provides a link with the Davenport–Hasse theorem on lifted Gauss sums.

The decompositions in our main theorem 5 corroborate results in the preprint by Gurevich and Howe [3] (which appeared after we finished our manuscript), especially those in Section 2.

2 Preliminaries

We consider the Weil representation and its character ω for $\mathrm{Sp}(2n, q)$, q a power of an odd prime, as a matrix group on $V = GF(q)^{2n}$ (with right action). We will be following the notation about ω in [13], [14] and use some explicit results in [14] about its values.

The symplectic form φ has matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Writing members of V as pairs (x, y) , with $x, y \in GF(q)^n$, one has

$$\varphi((x_1, y_1), (x_2, y_2)) = x_1 y_2^T - y_1 x_2^T,$$

the superscript T standing for transpose.

The subgroup G of $\mathrm{Sp}(2n, q)$ to be considered is $G = \langle -I \rangle \times B$, where B is the set of matrices $g_Q = \begin{bmatrix} I & 0 \\ Q & I \end{bmatrix}$. Such a matrix is in $\mathrm{Sp}(2n, q)$ just when $Q^T = Q$; that is, Q is symmetric (“ Q ” emphasizes the associated quadratic form $Q(v) = vQv^T$ on $GF(q)^n$). Let \mathbf{S}_n be the space of $n \times n$ symmetric matrices over $GF(q)$. There are $q^{n(n+1)/2}$ such matrices.

The following theorem follows from results in [2], but we present a proof done in the spirit of [9].

Theorem 1 *G is **Weil-free**: the irreducible constituents of $\omega|_G$ appear with multiplicity 1.*

Proof. We use the **orbit criterion**: an Abelian subgroup H of $\mathrm{Sp}(2n, q)$ is Weil-free exactly when the number of orbits of H on V is q^n [9]. To count the orbits of G , recall that $(\# \text{ orbits}) = |G|^{-1} \sum_{v \in V} |G_v|$, G_v the stabilizer of v . Notice that the matrices $-g_Q = \begin{bmatrix} -I & 0 \\ -Q & -I \end{bmatrix}$

in $-B$ fix only 0. Since $(x, y)^{g_Q} = (x, y) \begin{bmatrix} I & 0 \\ Q & I \end{bmatrix} = (x + yQ, y)$, $v = (x, y)$ is fixed exactly when $yQ = 0$. We look at three cases:

$v = (0, 0)$: then $|G_{(0,0)}| = |G| = 2q^{n(n+1)/2}$.

$v = (x, 0), x \neq 0$: all members of B fix v and $|G_{(x,0)}| = q^{n(n+1)/2}$. There are $q^n - 1$ such v .

$v = (x, y), y \neq 0$: v is fixed by the elements g_Q with $yQ = 0$. We can set up such Q by thinking of it as a quadratic form with y in the radical. Write $GF(q)^n = \langle y \rangle \oplus W$. Then Q can be given by taking a form on W and extending it by 0 on $\langle y \rangle$. That gives $|G_{(x,y)}| = q^{n(n-1)/2}$, the number of choices for the form on W . There are $q^n(q^n - 1)$ of these v .

Thus for $\sum_{v \in V} |G_v|$ we get

$$\begin{aligned} \sum_{v \in V} |G_v| &= 2q^{n(n+1)/2} + (q^n - 1) \times q^{n(n+1)/2} + q^n(q^n - 1) \times q^{n(n-1)/2} \\ &= q^{n(n+1)/2} \{2 + q^n - 1 + q^n - 1\} \\ &= q^n \times 2q^{n(n+1)/2}. \end{aligned}$$

So $|G|^{-1} \sum_{v \in V} |G_v| = q^n$, as needed. ■

3 Irreducible characters of G

To describe the characters of G , let ψ be the **canonical additive character** of $GF(q)$, as used in [13] (the terminology is that of [7, p. 190]): $\psi(\alpha) = e^{(2\pi i/p)\mathrm{tr}(\alpha)}$, where tr is the trace function $GF(q) \rightarrow GF(p)$, p the prime dividing q . Each linear character of the additive group of $GF(q)$ is given by $\alpha \rightarrow \psi(\beta\alpha)$, $\beta \in GF(q)$ [7, Theorem 5.7] (this is equivalent to the nondegeneracy of the trace form $(\alpha, \beta) \rightarrow \mathrm{tr}(\alpha\beta)$). In what follows, χ is the quadratic character on $GF(q)^\#$ (the nonzero elements) and $\delta = \chi(-1)$. In [13], ρ was defined as $\sum_{\alpha \in GF(q)} \psi(\alpha^2)$; we also have

$$\rho = \sum_{\beta \neq 0} \chi(\beta) \psi(\beta), \tag{1}$$

a Gaussian sum [7, Chapter 5, Section 2]; $\rho^2 = \delta q$. If $Q \in \mathbf{S}_n$, diagonalize Q and let

$$\Delta(Q) = \chi(\text{product of nonzero diagonal entries of } Q).$$

That is, if we write $GF(q)^n$ as $\mathrm{rad}(Q) \oplus W$, $\Delta(Q)$ is $\chi(\det(Q|W))$; $Q|W$ is the **nonsingular part** of Q . If Q has even rank $2k$, we call Q **hyperbolic** or **elliptic** according as $Q|W$ is hyperbolic or elliptic. If $Q|W$ is hyperbolic, then W is the orthogonal sum of hyperbolic planes, and $\det Q|W = (-1)^k$. Thus $\Delta(Q) = \delta^k$. If $Q|W$ is elliptic, then $\Delta(Q) = -\delta^k$.

Lemma 2 *The irreducible characters of B are the functions λ_S given by*

$$\lambda_S(g_Q) = \psi(\text{Tr}(SQ)),$$

where $S \in \mathbf{S}_n$ and Tr is the matrix trace. Each λ_S extends to two irreducible characters λ_S^\pm of G by the formula

$$\begin{aligned}\lambda_S^\pm(g_Q) &= \psi(\text{Tr}(SQ)) \\ \lambda_S^\pm(-g_Q) &= \pm\psi(\text{Tr}(SQ)),\end{aligned}\tag{2}$$

with the signs in the last equation matching on the two sides.

Proof. That this formula does give all the linear characters of B follows from the fact that the trace form $(S, Q) \rightarrow \text{Tr}(SQ)$ on \mathbf{S}_n is nondegenerate. That, in turn, can be seen as follows: suppose that $\text{Tr}(SQ) = 0$ for all $Q \in \mathbf{S}_n$. Take a basis for $GF(q)^n$ that makes S diagonal. Suppose that ζ is a nonzero diagonal entry of S . Choose Q to have 1 at that position and 0 elsewhere. Then $\text{Tr}(SQ) = \zeta$; so $\zeta = 0$ after all. Thus $S = 0$. So these characters λ_S are all distinct; and since there is the correct number of them, they give all the characters of B . Then for the characters of G , we use the direct product decomposition $G = \langle -I \rangle \times B$ to write them as claimed. ■

4 Values of ω on G

For the values of ω , we use results from [14]. A member $-g_Q$ of $-B$ has $-g_Q - 1$ invertible, with diagonal entries -2 , and [14, Section 6.5] gives

$$\omega(-g_Q) = \delta^n \chi(\det(-g_Q - 1)) = \delta^n \chi((-2)^{2n}) = \delta^n.\tag{3}$$

As for g_Q , [14, Theorem 6.7] implies that $\omega(g) = q^n \rho^{-\dim V^{g^{-1}}} \chi(\det \Theta_g)$. Here Θ_g is given in [14, Definition 3.3]: it is the form defined on $V^{g^{-1}}$ by $\Theta_g(u^{g^{-1}}, v^{g^{-1}}) = \varphi(u^{g^{-1}}, v)$. We have $(x, y)^{g^{-1}} = (yQ, 0)$. So

$$\begin{aligned}\Theta_g((x_1, y_1)^{g^{-1}}, (x_2, y_2)^{g^{-1}}) &= \varphi((x_1, y_1)^{g^{-1}}, (x_2, y_2)) \\ &= \varphi((y_1 Q, 0), (x_2, y_2)) \\ &= y_1 Q y_2^T.\end{aligned}$$

It follows that $\chi(\det \Theta_g) = \Delta(Q)$. Thus

Proposition 3 *The values of ω on G are given by*

$$\omega(g_Q) = q^n \rho^{-\text{rank } Q} \Delta(Q)\tag{4}$$

$$\omega(-g_Q) = \delta^n.\tag{5}$$

If $Q = 0$, $\Delta(Q)$ is artificially taken to be 1.

5 Character multiplicities in $\omega|G$

The multiplicity (which is 0 or 1) of a linear character λ in $\omega|G$ is $|G|^{-1} \sum_{g \in G} \omega(g) \overline{\lambda(g)}$. With $\lambda = \lambda_S^\pm$, we get

$$\sum_{g \in G} \omega(g) \overline{\lambda_S^\pm(g)} = \sum_{Q \in \mathcal{S}_n} q^n \rho^{-\text{rank} Q} \Delta(Q) \psi(-\text{Tr}(SQ)) \pm \sum_{Q \in \mathcal{S}_n} \delta^n \psi(-\text{Tr}(SQ))$$

(again, the sign on the right matches the sign in λ_S^\pm). The second summation is 0 if $S \neq 0$, and $\delta^n q^{n(n+1)/2}$ if $S = 0$.

For further computations, we need some standard group-order formulas. They are taken from [12].

Group	Order	
$GL(l, q)$	$q^{l(l-1)/2} \prod_{i=1}^l (q^i - 1)$	
$O^+(2k, q)$	$2q^{k(k-1)} (q^k - 1) \prod_{i=1}^{k-1} (q^{2i} - 1)$	(6)
$O^-(2k, q)$	$2q^{k(k-1)} (q^k + 1) \prod_{i=1}^{k-1} (q^{2i} - 1)$	
$O(2k+1, q)$	$2q^{k^2} \prod_{i=1}^k (q^{2i} - 1)$	

We also need the q -binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ that gives the number of r -dimensional subspaces of $GF(q)^n$. By duality, $\begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{bmatrix} n \\ n-r \end{bmatrix}_q$.

5.1 $S = 0$

When $S = 0$,

$$(\omega, \lambda_0^\pm)_G = \frac{1}{2q^{n(n-1)/2}} \sum_{Q \in \mathcal{S}_n} \rho^{-\text{rank} Q} \Delta(Q) \pm \frac{\delta^n}{2},$$

since $|G| = 2q^{n(n+1)/2}$. We conclude that the first term must be $1/2$:

$$\sum_{Q \in \mathcal{S}_n} \rho^{-\text{rank} Q} \Delta(Q) = q^{n(n-1)/2}. \quad (7)$$

We shall elaborate on this in Section 8. Thus we have

Proposition 4 *The multiplicity of λ_0^\pm in ω is*

$$(\omega, \lambda_0^\pm)_G = \frac{1 \pm \delta^n}{2}.$$

5.2 $S \neq 0$

Here

$$\sum_{g \in G} \omega(g) \overline{\lambda_S^\pm(g)} = \sum_{Q \in \mathbf{S}_n} q^n \rho^{-\text{rank} Q} \Delta(Q) \psi(-\text{Tr}(SQ)).$$

It follows that λ_S^+ and λ_S^- appear with the same multiplicity in $\omega|G$. Now note that $\text{Tr}(M^T S M Q) = \text{Tr}(S M Q M^T)$. So with M nonsingular, λ_S^\pm and $\lambda_{M^T S M}^\pm$ also have the same multiplicity in $\omega|G$. Suppose that $\text{rank} S = r$. The number of members of \mathbf{S}_n congruent to S (that is, of the form $M^T S M$) is

$$\begin{bmatrix} n \\ n-r \end{bmatrix}_q \times \frac{|GL(r, q)|}{|O(S)|} = \begin{bmatrix} n \\ r \end{bmatrix}_q \times \frac{|GL(r, q)|}{|O(S)|},$$

where $O(S)$ is the orthogonal group for S . We want to show that if $r > 1$, then this number is more than $(q^n - 1)/2$. That will imply that the only $S \neq 0$ that can appear in the characters in $\omega|G$ are the ones of rank 1; there are $q^n - 1$ of these [8, Theorem 13.2.47]. Again we separate by parity.

- $r = 2k$: Then

$$|O(S)| \leq |O^-(2k, q)| = 2q^{k(k-1)}(q^k + 1) \prod_{i=1}^{k-1} (q^{2i} - 1),$$

from (6)

So

$$\begin{aligned} \begin{bmatrix} n \\ r \end{bmatrix}_q \times \frac{|GL(r, q)|}{|O(S)|} &\geq \begin{bmatrix} n \\ 2k \end{bmatrix}_q \times \frac{|GL(2k, q)|}{|O^-(2k, q)|} \\ &= \frac{\prod_{j=n-2k+1}^n (q^j - 1)}{\prod_{j=1}^{2k} (q^j - 1)} \times \frac{q^{k(2k-1)} \prod_{j=1}^{2k} (q^j - 1)}{2q^{k(k-1)}(q^k + 1) \prod_{i=1}^{k-1} (q^{2i} - 1)} \\ &= \frac{\prod_{j=n-2k+1}^n (q^j - 1)}{2(q^k + 1) \prod_{i=1}^{k-1} (q^{2i} - 1)} \times q^{k^2} \\ &\geq \frac{\prod_{j=n-2k+1}^n (q^j - 1)}{2(q^k + 1) \prod_{i=1}^{k-1} q^{2i}} \times q^{k^2} \\ &= \frac{\prod_{j=n-2k+1}^n (q^j - 1)}{2(q^k + 1)} \times q^k. \end{aligned}$$

We would like this to be more than $(q^n - 1)/2$, that is,

$$\frac{\prod_{j=n-2k+1}^n (q^j - 1)}{2(q^k + 1)} \times q^k > \frac{q^n - 1}{2}.$$

We need

$$\prod_{j=n-2k+1}^{n-1} (q^j - 1) > 1 + \frac{1}{q^k}.$$

Since $n - 1 \geq n - 2k + 1$, from $k \geq 1$, and $n - 2k + 1 \geq 1$, the product is nonempty and its smallest factor is at least 2. Thus the inequality holds.

- $r = 2k + 1$: This time, again with the appropriate formula from (6) filled in,

$$\begin{aligned} \begin{bmatrix} n \\ r \end{bmatrix}_q \times \frac{|GL(r, q)|}{|O(S)|} &= \frac{\prod_{j=n-2k}^n (q^j - 1)}{\prod_{j=1}^{2k+1} (q^j - 1)} \times \frac{q^{k(2k+1)} \prod_{j=1}^{2k+1} (q^j - 1)}{2q^{k^2} \prod_{j=1}^k (q^{2j} - 1)} \\ &= \frac{\prod_{j=n-2k}^n (q^j - 1)}{2 \prod_{j=1}^k (q^{2j} - 1)} \times q^{k^2+k} \\ &\geq \frac{\prod_{j=n-2k}^n (q^j - 1)}{2 \prod_{j=1}^k q^{2j}} \times q^{k^2+k} \\ &= \frac{\prod_{j=n-2k}^n (q^j - 1)}{2}. \end{aligned}$$

We would also like this to be greater than $(q^n - 1)/2$, and as long as $k > 0$, it is.

6 The decomposition of $\omega|G$

Collecting the results of the preceding section gives

$$\omega|G = \frac{1 + \delta^n}{2} \lambda_0^+ + \frac{1 - \delta^n}{2} \lambda_0^- + \sum_{\text{rank } S=1} \left(\frac{1}{2q^{n(n-1)/2}} \sum_{Q \in \mathcal{S}_n} \rho^{-\text{rank } Q} \Delta(Q) \psi(-\text{Tr}(SQ)) \right) (\lambda_S^+ + \lambda_S^-).$$

We still need to determine which congruence class of symmetric matrices S of rank 1 actually appears in the decomposition. (There are two such classes, corresponding to $\Delta(S) = 1$ and $\Delta(S) = -1$. Each class has $(q^n - 1)/2$ members.) To do so, we examine the characters on a small subgroup of G .

Let M be the $n \times n$ symmetric matrix with $M_{11} = 1$ and all other entries 0; $\Delta(M) = 1$. Let H be the subgroup consisting of the matrices $h_\alpha = \begin{bmatrix} I & 0 \\ \alpha M & I \end{bmatrix}$, $\alpha \in GF(q)$. Then $\omega(h_\alpha) = q^n$ if $\alpha = 0$, and $\omega(h_\alpha) = q^n \rho^{-1} \chi(\alpha)$ if $\alpha \neq 0$, by (4). Moreover, $\lambda_{\beta M}^\pm(h_\alpha) = \psi(\alpha\beta)$, for $\beta \neq 0$. It follows that

$$\begin{aligned} (\omega, \lambda_{\beta M}^\pm)_H &= q^{-1} \left\{ q^n + \sum_{\alpha \neq 0} q^n \rho^{-1} \chi(\alpha) \psi(-\alpha\beta) \right\} \\ &= q^{n-1} \left\{ 1 + \delta \rho^{-1} \chi(\beta) \sum_{\alpha \neq 0} \chi(-\beta\alpha) \psi(-\beta\alpha) \right\} \\ &= q^{n-1} (1 + \chi(\beta) \delta), \end{aligned}$$

by (1). So $\lambda_{\beta M}^\pm$ appears in $\omega|H$ just when $\chi(\beta) = \delta$. This implies the following:

Theorem 5

$$\omega|G = \frac{1 + \delta^n}{2} \lambda_0^+ + \frac{1 - \delta^n}{2} \lambda_0^- + \sum_{\substack{\text{rank } S=1 \\ \Delta(S)=\delta}} (\lambda_S^+ + \lambda_S^-). \quad (8)$$

In particular,

$$\omega|B = \lambda_0 + 2 \sum_{\substack{\text{rank } S=1 \\ \Delta(S)=\delta}} \lambda_S. \quad (9)$$

The Weil character is the sum of two irreducible characters, ω_+ , of degree $(q^n + 1)/2$, and ω_- , of degree $(q^n - 1)/2$. Their values at $-I$ are $\omega_\pm(-I) = \pm \delta^n (q^n \pm 1)/2$ [14, Section 6], the signs all matching. Observing the eigenvalues of $-I$ in the corresponding representations, we can write that

$$\begin{aligned} \omega_+|G &= \frac{1 + \delta^n}{2} \lambda_0^+ + \frac{1 - \delta^n}{2} \lambda_0^- + \sum_{\substack{\text{rank } S=1 \\ \Delta(S)=\delta}} \lambda_S^{\delta^n} \\ \omega_-|G &= \sum_{\substack{\text{rank } S=1 \\ \Delta(S)=\delta}} \lambda_S^{-\delta^n}. \end{aligned}$$

As an immediate consequence we get that both $\omega_+|B$ and $\omega_-|B$ are multiplicity free.

7 Confirmation of the $\omega|B$ decomposition

Recall that $g_Q = \begin{bmatrix} I & 0 \\ Q & I \end{bmatrix}$. By (4), $\omega(g_Q) = q^n \rho^{-\text{rank} Q} \Delta(Q)$. Let $\text{rank} Q = r$. Then (9) gives

$$\omega(g_Q) = q^n \rho^{-r} \Delta(Q) = 1 + 2 \sum_{\substack{\text{rank} S=1 \\ \Delta(S)=\delta}} \psi(\text{Tr}(SQ)). \quad (10)$$

If $x \in GF(q)^n$, $x \neq 0$, then $x^T x$ is a rank 1 symmetric matrix. Two such products $x^T x$ and $y^T y$ are equal just when $y = \pm x$. For $x = (\xi_1, \dots, \xi_n)$, the diagonal entries of $x^T x$ are the ξ_i^2 . Thus since at least one is nonzero, $\Delta(x^T x) = 1$. So all symmetric $n \times n$ rank 1 matrices can be written as $x^T x$ ($x \neq 0$) or $\nu x^T x$, where ν is a fixed nonsquare in $GF(q)$. (As mentioned above, there are $(q^n - 1)/2$ matrices of each type.) We can rewrite (10) as follows:

$$\omega(g_Q) = q^n \rho^{-r} \Delta(Q) = \sum_x \left\{ \frac{1+\delta}{2} \psi(\text{Tr}(x^T x Q)) + \frac{1-\delta}{2} \psi(\text{Tr}(\nu x^T x Q)) \right\}. \quad (11)$$

The factors $(1 \pm \delta)/2$ pick out the S with $\Delta(S) = \delta$ and adjust for the fact that each S appears twice; the 1 on the right in (10) comes from $x = 0$. (Recall also that we have set $\Delta(0) = 1$.)

Now

$$\text{Tr}(x^T x Q) = \text{Tr}(x Q x^T) = \text{Tr}(Q(x)) = Q(x),$$

so the preceding formula becomes

$$q^n \rho^{-r} \Delta(Q) = \sum_x \left\{ \frac{1+\delta}{2} \psi(Q(x)) + \frac{1-\delta}{2} \psi(\nu Q(x)) \right\}. \quad (12)$$

Let σ denote the sum. To evaluate σ we need the number of times $Q(x) = \alpha$ for $\alpha \in GF(q)$. These counts for nonzero α depend only on whether α is a square or not, since $Q(\beta x) = \beta^2 Q(x)$. Let 0 be taken on Z times (including $Q(0) = 0$); a given nonzero square S times; and a given nonsquare N times. Then $Z + (S + N)(q - 1)/2 = q^n$.

We also need the character sums

$$\sum_{\alpha \neq 0 \text{ square}} \psi(\alpha) = \frac{\rho - 1}{2} \text{ and } \sum_{\alpha \text{ nonsquare}} \psi(\alpha) = \frac{-\rho - 1}{2}, \quad (13)$$

which follow from

$$\sum_{\alpha \neq 0} \psi(\alpha) = -1, \quad \sum_{\alpha \neq 0 \text{ square}} \psi(\alpha) - \sum_{\alpha \text{ nonsquare}} \psi(\alpha) = \rho.$$

Collect terms in σ according to $Q(x)$:

$$\begin{aligned}\sigma &= Z \left\{ \frac{1+\delta}{2} + \frac{1-\delta}{2} \right\} + S \left\{ \frac{1+\delta}{2} \frac{\rho-1}{2} + \frac{1-\delta}{2} \frac{-\rho-1}{2} \right\} \\ &\quad + N \left\{ \frac{1+\delta}{2} \frac{-\rho-1}{2} + \frac{1-\delta}{2} \frac{\rho-1}{2} \right\} \\ &= Z - \frac{S+N}{2} + \delta \rho \frac{S-N}{2}.\end{aligned}$$

Then put $Z = q^n - (S+N)(q-1)/2$ to obtain

$$\sigma = q^n - \frac{q(S+N)}{2} + \delta \rho \frac{S-N}{2}. \quad (14)$$

Now we need S and N . If Q_0 is the nonsingular part of Q , the counts for Q are those for Q_0 multiplied by q^{n-r} . Here are these numbers for Q , obtained from [7, Theorems 6.26 and 6.27] for Q_0 :

$$\begin{array}{ll} r \text{ even:} & \overset{S}{q^{n-1} - \delta^{r/2} \Delta(Q) q^{n-r/2-1}} \quad \overset{N}{q^{n-1} - \delta^{r/2} \Delta(Q) q^{n-r/2-1}} = S \\ r \text{ odd:} & q^{n-1} + \delta^{(r-1)/2} q^{n-(r+1)/2} \Delta(Q) \quad q^{n-1} - \delta^{(r-1)/2} q^{n-(r+1)/2} \Delta(Q). \end{array} \quad (15)$$

Substituting into (14), we obtain simplifications corresponding to the parity of r .

- r even: then $S = N$ and

$$\begin{aligned}\sigma &= q^n - \frac{q}{2} (2q^{n-1} - 2\delta^{r/2} \Delta(Q) q^{n-r/2-1}) \\ &= \delta^{r/2} \Delta(Q) q^{n-r/2}.\end{aligned}$$

Since $q^{r/2} = \delta^{r/2} \rho^r$, this is correctly $q^n \rho^{-r} \Delta(Q)$.

- r odd: then

$$\begin{aligned}\sigma &= q^n - \frac{q}{2} 2q^{n-1} + \delta^{(r-1)/2} q^{n-(r+1)/2} \Delta(Q) \delta \rho \\ &= \delta^{(r+1)/2} q^n q^{-(r+1)/2} \rho \Delta(Q) \\ &= \delta^{(r+1)/2} q^n \delta^{(r+1)/2} \rho^{-r-1} \rho \Delta(Q) \\ &= q^n \rho^{-r} \Delta(Q),\end{aligned}$$

again correct.

7.1 The S, N formulas

In point of fact, the formulas for S and N follow from those for $\omega|B$. Combining (10) and (12) gives

$$q^n \rho^{-r} \Delta(Q) = q^n - \frac{q(S+N)}{2} + \delta \rho \frac{S-N}{2}. \quad (16)$$

For a needed second equation, let Q' be Q scaled by a nonsquare. Then the counts for Q' are $S' = N$ and $N' = S$, and $\Delta(Q') = (-1)^r \Delta(Q)$. Formula (16) for Q' reads

$$q^n \rho^{-r} (-1)^r \Delta(Q) = q^n - \frac{q(S+N)}{2} + \delta \rho \frac{N-S}{2}. \quad (17)$$

Solving (16) and (17) for S and N produces

$$\begin{aligned} S &= q^{n-1} - \frac{1+(-1)^r}{2} q^{n-1} \rho^{-r} \Delta(Q) + \frac{1-(-1)^r}{2} q^n \rho^{-r-1} \delta \Delta(Q) \\ N &= q^{n-1} - \frac{1+(-1)^r}{2} q^{n-1} \rho^{-r} \Delta(Q) - \frac{1-(-1)^r}{2} q^n \rho^{-r-1} \delta \Delta(Q), \end{aligned}$$

and then

$$\begin{aligned} Z &= q^n - (S+N) \frac{q-1}{2} \\ &= q^{n-1} + \frac{1+(-1)^r}{2} q^{n-1} (q-1) \rho^{-r} \Delta(Q). \end{aligned}$$

These are uniform expressions (perhaps not obvious in traditional derivations!) which give (15) on taking r even or odd.

Incidentally, when $\delta = -1$ or q is not a square, ρ is not rational. In that case, the formulas follow from equating coefficients in (16) for the field $\mathbb{Q}(\rho) = \mathbb{Q} + \mathbb{Q}\rho$ and again solving for S and N .

8 A q -binomial identity

Recall equation (7):

$$\sum_{Q \in \mathcal{S}_n} \rho^{-\text{rank} Q} \Delta(Q) = q^{n(n-1)/2}.$$

If $\text{rank} Q$ is odd, then with ν a nonsquare in $GF(q)$, $\Delta(\nu Q) = -\Delta(Q)$, and the terms for Q and νQ in the sum cancel. Thus as $\rho^{-2k} = \delta^k q^{-k}$,

$$q^{n(n-1)/2} = \sum_{Q \in \mathcal{S}_n} \rho^{-\text{rank} Q} \Delta(Q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{Q \in \mathcal{S}_n \\ \text{rank} Q = 2k}} \delta^k q^{-k} \Delta(Q). \quad (18)$$

Let $\text{rank} Q = 2k$. As pointed out in Section 3, if Q is hyperbolic, then $\Delta(Q) = \delta^k$; and if Q is elliptic, then $\Delta(Q) = -\delta^k$.

Now we can give a specific formula for $\sum_{\substack{Q \in \mathbf{S}_n \\ \text{rank} Q = 2k}} \delta^k q^{-k} \Delta(Q)$. The number of terms with Q hyperbolic is

$$\begin{bmatrix} n \\ 2k \end{bmatrix}_q \times \frac{|GL(2k, q)|}{|O^+(2k, q)|},$$

and the number with Q elliptic is

$$\begin{bmatrix} n \\ 2k \end{bmatrix}_q \times \frac{|GL(2k, q)|}{|O^-(2k, q)|}.$$

Thus

$$\sum_{\substack{Q \in \mathbf{S}_n \\ \text{rank} Q = 2k}} \delta^k q^{-k} \Delta(Q) = \delta^k q^{-k} \begin{bmatrix} n \\ 2k \end{bmatrix}_q \times |GL(2k, q)| \times \delta^k \left(\frac{1}{|O^+(2k, q)|} - \frac{1}{|O^-(2k, q)|} \right);$$

the second δ^k is the factor needed for $\Delta(Q)$. By (6),

$$\frac{1}{|O^+(2k, q)|} - \frac{1}{|O^-(2k, q)|} = \frac{1}{q^{k(k-1)} \prod_{i=1}^k (q^{2i} - 1)}$$

(note the addition of one more factor in the product). So, again by (6),

$$\begin{aligned} \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{Q \in \mathbf{S}_n \\ \text{rank} Q = 2k}} \delta^k q^{-k} \Delta(Q) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \delta^k q^{-k} \begin{bmatrix} n \\ 2k \end{bmatrix}_q \times \delta^k \frac{q^{k(2k-1)} \prod_{i=1}^{2k} (q^i - 1)}{q^{k(k-1)} \prod_{i=1}^k (q^{2i} - 1)} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{k^2-k} \prod_{i=0}^{k-1} (q^{2i+1} - 1). \end{aligned}$$

Thus from (18),

$$q^{n(n-1)/2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{k^2-k} \prod_{i=0}^{k-1} (q^{2i+1} - 1).$$

Is that really true? Yes – it gives the count of the number of symplectic (skew-symmetric) $n \times n$ matrices over $GF(q)$. From [12], $|Sp(2k, q)| = q^{k^2} \prod_{i=1}^k (q^{2i} - 1)$, so the number of

symplectic matrices of rank $2k$ is

$$\begin{aligned} \begin{bmatrix} n \\ 2k \end{bmatrix}_q \times \frac{|GL(2k, q)|}{|Sp(2k, q)|} &= \begin{bmatrix} n \\ 2k \end{bmatrix}_q \frac{q^{k(2k-1)} \prod_{i=1}^{2k} (q^i - 1)}{q^{k^2} \prod_{i=1}^k (q^{2i} - 1)} \\ &= \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{k^2-k} \prod_{i=0}^{k-1} (q^{2i+1} - 1) \end{aligned}$$

[8, Theorem 13.2.48], and that is to be summed from $k = 0$ to $\lfloor n/2 \rfloor$ (giving 1 at $k = 0$). But the total number of symplectic $n \times n$ matrices is simply $q^{n(n-1)/2}$.

9 Minimum Weil-free subgroups of G

Suppose that H is a subgroup of G . If $\omega|_H$ is also Weil-free, then $|H| \geq q^n$. It cannot be that $|H| = q^n$, because then $\omega|_H$ would just be the sum of all q^n linear characters of H . But that sum is q^n at I and 0 at $h \neq I$, whereas $\omega(h) \neq 0$, by (4). Thus $|H| \geq 2q^n$. We shall show that there are Weil-free subgroups of order $2q^n$. If $|H| = 2q^n$, then $H = \langle -I \rangle \times (H \cap B)$.

Adapting the orbit count in the proof of Theorem 1, we find that $2q^n \times (\text{number of orbits})$ of such an H is

$$2q^n + (q^n - 1)q^n + \sum_x \sum_{y \neq 0} |H_{(x,y)}| \geq 2q^n + (q^n - 1)q^n + q^n(q^n - 1) = 2q^{2n}.$$

So if H is to be Weil-free, each $H_{(x,y)}$ with $y \neq 0$ must be just $\{I\}$. Again as in the proof of Theorem 1, this means that if $g_Q \in H$, with $Q \neq 0$, then Q must have full rank n . So what would work is an n -dimensional subspace W of \mathcal{S}_n whose nonzero members are all nonsingular. Then $H \cap B$ would be $\{g_Q | Q \in W\}$.

To construct W , realize $GF(q)^n$ as $GF(q^n)$ and let tr be the trace function $GF(q^n) \rightarrow GF(q)$. Then for $\alpha \in GF(q^n)$, the function Q_α given by $Q_\alpha(\zeta) = \text{tr}(\alpha\zeta^2)$ is a quadratic form on $GF(q^n)$. The corresponding bilinear form is $B_\alpha(\xi, \eta) = \text{tr}(\alpha\xi\eta)$. This is nondegenerate when $\alpha \neq 0$, making Q_α nonsingular then. Now let $W = \{Q_\alpha | \alpha \in GF(q^n)\}$.

9.1 An evaluation for H

Formula (12) for Q_a reads

$$q^n \rho^{-n} \Delta(Q_\alpha) = \sum_{\zeta \in GF(q^n)} \left\{ \frac{1+\delta}{2} \psi(Q_\alpha(\zeta)) + \frac{1-\delta}{2} \psi(\nu Q_\alpha(\zeta)) \right\}.$$

Because $\nu \in GF(q)$ and $\rho^2 = \delta q$, this becomes

$$\delta^n \rho^n \Delta(Q_\alpha) = \sum_{\zeta \in GF(q^n)} \left\{ \frac{1+\delta}{2} \psi(\text{tr}(\alpha\zeta^2)) + \frac{1-\delta}{2} \psi(\text{tr}(\nu\alpha\zeta^2)) \right\}. \quad (19)$$

One has to be careful with the matrix interpretation. Let ζ_1, \dots, ζ_n be a $GF(q)$ -basis of $GF(q^n)$. Then the matrix for B_α is $[\text{tr}(\alpha\zeta_i\zeta_j)]$. This can be written as

$$\begin{aligned} [\text{tr}(\alpha\zeta_i\zeta_j)] &= \begin{bmatrix} \zeta_1 & \zeta_1^q & \dots & \zeta_1^{q^{n-1}} \\ \zeta_2 & \zeta_2^q & \dots & \zeta_2^{q^{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ \zeta_n & \zeta_n^q & \dots & \zeta_n^{q^{n-1}} \end{bmatrix} \begin{bmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha^q & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \alpha^{q^{n-1}} \end{bmatrix} \begin{bmatrix} \zeta_1 & \zeta_2 & \dots & \zeta_n \\ \zeta_1^q & \zeta_2^q & \dots & \zeta_n^q \\ \vdots & \vdots & \dots & \vdots \\ \zeta_1^{q^{n-1}} & \zeta_2^{q^{n-1}} & \dots & \zeta_n^{q^{n-1}} \end{bmatrix} \\ &= D \times \text{diag}(\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}) \times D^T. \end{aligned}$$

Taking determinants gives

$$\det [\text{tr}(\alpha\zeta_i\zeta_j)] = (\det D)^2 \prod_{i=0}^{n-1} \alpha^{q^i} = (\det D)^2 N(\alpha),$$

in which $(\det D)^2$ is the discriminant of the extension $GF(q^n)/GF(q)$ and $\prod_{i=0}^{n-1} \alpha^{q^i}$ is the norm $N(\alpha)$ of α . Then

$$\Delta(Q_\alpha) = \chi((\det D)^2) \chi(N(\alpha)).$$

Applying the automorphism $\xi \rightarrow \xi^q$ to D cycles its columns; so $(\det D)^q = (-1)^{n-1} \det D$, the sign being that of an n -cycle. Thus $\chi((\det D)^2) = (-1)^{n-1}$: $(\det D)^2$ is a square in $GF(q)$ only when its square-root $\det D$ is in $GF(q)$! For $\chi(N(\alpha))$ we have

$$\chi(N(\alpha)) = N(\alpha)^{\frac{q-1}{2}} = \left(\alpha^{\frac{q^n-1}{q-1}} \right)^{\frac{q-1}{2}} = \alpha^{\frac{q^n-1}{2}} = X(\alpha),$$

where $X(\alpha)$ is the quadratic character of α for the field $GF(q^n)$ (we can determine any $\chi(z)$ by reading it in $GF(q)$). All together, $\Delta(Q_\alpha) = (-1)^{n-1} X(\alpha)$.

For the right side of (19), we have that $\sum_{\zeta} \psi(\text{tr}(\beta\zeta^2)) = X(\beta)P$, where P is the “ ρ ” for $GF(q^n)$, by the formulas in (13) for $GF(q^n)$. Moreover, since ν is a nonsquare in $GF(q)$, $X(\nu) = (-1)^n$. Therefore (19) becomes

$$\begin{aligned} \delta^n \rho^n (-1)^{n-1} X(\alpha) &= \left\{ \frac{1+\delta}{2} X(\alpha)P + \frac{1-\delta}{2} (-1)^n X(\alpha)P \right\} \\ &= X(\alpha)P \left\{ \frac{1+(-1)^n}{2} + \delta \frac{1-(-1)^n}{2} \right\}, \end{aligned}$$

or

$$\delta^n \rho^n (-1)^{n-1} = P \left\{ \frac{1+(-1)^n}{2} + \delta \frac{1-(-1)^n}{2} \right\}.$$

This simplifies to

$$P = (-1)^{n-1} \rho^n,$$

on sorting by the parity of n . That is a particular instance of the Davenport-Hasse theorem on lifted Gauss sums [7, Theorem 5.14].

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